

PARTIAL ISOMETRIES OF A SUB-RIEMANNIAN MANIFOLD

MAHUYA DATTA

ABSTRACT. In this article, we obtain the following generalisation of isometric C^1 -immersion theorem of Nash and Kuiper. Let M be a smooth manifold of dimension m and H a rank k subbundle of the tangent bundle TM with a Riemannian metric g_H . Then the pair (H, g_H) defines a sub-Riemannian structure on M . We call a C^1 -map $f : (M, H, g_H) \rightarrow (N, h)$ into a Riemannian manifold (N, h) a *partial isometry* if the derivative map df restricted to H is isometric, that is if $f^*h|_H = g_H$. We prove that if $f_0 : M \rightarrow N$ is a smooth map such that $df_0|_H$ is a bundle monomorphism and $f_0^*h|_H < g_H$, then f_0 can be homotoped to a C^1 -map $f : M \rightarrow N$ which is a partial isometry, provided $\dim N > k$. As a consequence of this result, we obtain that every sub-Riemannian manifold (M, H, g_H) admits a partial isometry in \mathbb{R}^n , provided $n \geq m + k$.

Key words: Sub-Riemannian manifold, partial isometry, convex integration.

Mathematics Subject Classification 2000: 53C17, 58J99.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold and $f_0 : M \rightarrow \mathbb{R}^n$ be a C^∞ map such that $f_0^*h < g$ (that is, $g - f_0^*h$ is positive definite), where h is the canonical metric on the Euclidean space \mathbb{R}^n . Nash proved in [9] that if f_0 is an immersion (respectively an embedding) then f_0 can be homotoped to an isometric immersion (respectively embedding) $f : M \rightarrow \mathbb{R}^n$ so that $f^*h = g$, provided $n \geq \dim M + 2$. He further observed that a closed manifold M that immerses (respectively embeds) in \mathbb{R}^n also does so isometrically under the same dimension restriction. Shortly after this, Kuiper [8] proved that these results are true even when $n \geq \dim M + 1$. By Whitney's Immersion Theorem it is known that every manifold M of dimension m admits an immersion in \mathbb{R}^{2m} and therefore, it admits an isometric C^1 immersion by Nash-Kuiper theorem. Isometric immersions $f : (M, g) \rightarrow (N, h)$ into any Riemannian manifold (N, h) of dimension n can be (locally) seen as solutions to a system of $m(m+1)/2$ equations in n variables, which is clearly overdetermined when $n < m(m+1)/2$. Therefore, for sufficiently large m , the system remains overdetermined for $n \geq 2m$. A remarkable aspect of Nash-Kuiper theorem is in showing that a overdetermined system may not only be solvable but the solution space can be 'very large'.

In this paper, we obtain a generalisation of the Nash-Kuiper isometric C^1 -immersion theorem which comes in response to certain observations of Gromov in [5, 2.4.9(B)]. Let M be a smooth manifold of dimension m and H a rank k subbundle of the tangent bundle TM with a Riemannian metric g_H . Then the pair (H, g_H) defines a sub-Riemannian structure on M [6]. We call a C^1 -map $f : (M, H, g_H) \rightarrow (N, h)$ into a Riemannian manifold (N, h) a *partial isometry* if

$df|_H$ is isometric, that is, if $f^*h|_H = g_H$. In the special situation, when H is an integrable distribution, we obtain a regular foliation \mathcal{F} on M such that $T\mathcal{F} = H$. The leaves of this foliation, being integral submanifolds of H , inherit Riemannian structures from the metric g_H on H . Therefore, a partial isometry in this case can be viewed as a C^1 map which restricts to an isometric immersion on each leaf of the foliation \mathcal{F} .

Partial isometries are also related to Carnot-Caratheodory geometry underlying the sub-Riemannian structure (M, H) . Let d_H denote the Carnot Caratheodory metric on M associated with the subbundle H of TM . Then for any two points x, y of M , $d_H(x, y) = \infty$ if there is no H -horizontal path in M connecting these points. Otherwise $d_H(x, y)$ is the infimum of the lengths of all H -horizontal paths between x and y . Recall that a piecewise smooth path $\gamma : I \rightarrow M$ is called H -horizontal if the tangent vectors $\dot{\gamma}(t)$ lies in H for all those $t \in I$ where the path is differentiable. Observe that a partial isometry preserves the norm of any vector in H , and hence preserves the lengths of H -horizontal paths in M . Thus, if $f : (M, g_H) \rightarrow (N, h)$ is a partial isometry then $f : (M, d_H) \rightarrow (N, d_h)$ is necessarily a *path-isometry*, where d_h is the intrinsic metric on N defined by h .

The main result of the paper may be stated as follows:

Theorem 1.1. *Let M be a manifold with a sub-Riemannian structure (H, g_H) defined as above and $f_0 : M \rightarrow N$ be a C^∞ map into a Riemannian manifold (N, h) satisfying the following conditions:*

- (i) *The restriction of df_0 to the bundle H is a monomorphism and*
- (ii) *$g_H - f_0^*h|_H$ is positive definite on H .*

If $\dim N > \text{rank } H$ then f_0 can be homotoped to a partial isometry $f : (M, g_H) \rightarrow (N, h)$. Furthermore, the homotopy can be made to lie in a given neighbourhood of f_0 in the fine C^0 topology.

If we take $H = TM$ in Theorem 1.1 then we obtain the Nash-Kuiper isometric C^1 -immersion theorem. Taking N to be an Euclidean space we prove the existence of partial isometries.

Corollary 1.2. *Every sub-Riemannian manifold (M, H, g_H) admits a partial isometry in \mathbb{R}^n provided $n \geq \dim M + \text{rank } H$.*

We also discuss several other consequences of Theorem 1.1 in Corollaries 4.6 and 5.1.

We use the convex integration technique (see [5], [4]) to prove the main theorem of this paper. It would be appropriate to mention here that Gromov developed the convex integration theory on the foundation of Kuiper's technique [8] and applied this theory to solve many interesting problems which appear in the context of geometry.

We organise the paper as follows. In Section 2, we outline the proof of Theorem 1.1, and in Section 3 we briefly discuss the convex integration technique following the beautiful exposition of Eliashberg and Mishachev [4]. In Section 4 we prove the main results of the paper stated above and in Section 5 we discuss some applications of Theorem 1.1.

2. SKETCH OF THE PROOF

Let (N, h) and (M, H, g_H) be as in Section 1 and let g_0 be a fixed Riemannian metric on M such that $g_0|_H = g_H$.

Definition 2.1. A C^1 map from M to N is called an H -immersion if its derivative restricts to a monomorphism on H (We have borrowed this terminology from [2]).

A C^1 map $f_0 : M \rightarrow N$ is said to be g_H -short if $g_H - f_0^*h$ restricted to H is positive definite. We use the notation $f_0^*h|_H < g_H$ to express g_H -shortness of f_0 .

It is easy to see that if f is an H -immersion then $f^*h|_H$ is a Riemannian metric on H and conversely. Also note that if f is a partial isometry then it is necessarily an H -immersion.

We now introduce two real-valued functions on M . The first function will measure the ‘norm’ of a bilinear form on H relative to a Riemannian metric on H . The second function will measure the ‘distance’ between two C^1 -functions relative to Riemannian metrics on M and N .

Let g be a Riemannian metric on H . For any bilinear form \bar{g} we define a function $n_g(\bar{g}) : M \rightarrow \mathbb{R}$, as follows:

$$n_g(\bar{g})(x) = \sup_{v \in H_x \setminus 0} \frac{|\bar{g}_x(v, v)|}{g_x(v, v)}$$

Given any Riemannian metric \tilde{g} on M and any two C^1 maps $f, \bar{f} : M \rightarrow N$, define a function $d_{\tilde{g}}(f, \bar{f}) : M \rightarrow \mathbb{R}$ by

$$d_{\tilde{g}}(f, \bar{f})(x) = \sup_{v \in T_x M \setminus 0} \frac{\|df_x(v) - d\bar{f}_x(v)\|_h}{\|v\|_{\tilde{g}}}$$

To simplify notations we shall denote n_{g_H} by n and d_{g_0} by d .

We now outline the proof of Theorem 1.1. We start with an f_0 as in the hypothesis of the theorem. First note that with the newly introduced terminology the hypothesis on f_0 reads as follows: (i) f_0 is an H -immersion, and (ii) f_0 is g_H -short. (The assumption of g_H -shortness is not necessary if M is a closed manifold and N is an Euclidean space: For, by multiplying a given H -immersion by a suitable positive scalar λ we can make it g_H -short.) The shortness condition means that $g_H - f_0^*h$ is a Riemannian metric on H . As in the proof of isometric C^1 -immersion theorem in [9], we need a suitable decomposition of $g_H - f_0^*h|_H$ on H .

Lemma 2.2. Let $\{U_\lambda | \lambda \in \Lambda\}$ be an open covering of the manifold M such that (a) each U_λ is a coordinate neighbourhood in M and (b) for any λ_0 , U_{λ_0} intersects at most $c_1(m)$ many U_λ ’s including itself. Then $g_H - f_0^*h|_H$ admits a decomposition as follows:

$$g_H - f_0^*h|_H = \sum_{i=1}^{\infty} \phi_i^2 (d\psi_i)^2|_H,$$

where ϕ_i and ψ_i , for $i = 1, 2, \dots$, are C^∞ functions on M such that

- (i) for each i there exists a $\lambda \in \Lambda$ for which $\text{supp } \phi_i$ is contained in U_λ and $d\psi_i$ is a rank 1 quadratic form on U_λ .
- (ii) for all but finitely many i , ϕ_i vanishes on an U_λ and
- (iii) there are at most $c(m)$ many ϕ_i ’s which are non-vanishing at any point x .

Here $c(m)$ and $c_1(m)$ are positive integers depending on $m = \dim M$.

Proof. Choose a subbundle K of TM which is complementary to H and take any Riemannian metric g_K on it. Then $g_M = (g_H - f_0^*h|_H) \oplus g_K + f_0^*h$ is a Riemannian metric on TM which clearly restricts to g_H on H . Moreover, $g_M - f_0^*h > 0$ on

M . Then by Nash's decomposition formula (see [1] and [9]), there exist smooth functions ψ_i and ϕ_i as described in the lemma such that $g_M - f_0^*h = \sum_i \phi_i^2 d\psi_i^2$. By restricting both sides to H we get the desired decomposition. \square

Construction of an Approximate solution: Applying Lemma 2.2 we get a decomposition of $g_H - f_0^*h$. We then use this decomposition to obtain a C^∞ map \tilde{f} which is very close to a partial isometry in the sense that $n(g_H - \tilde{f}^*h)$ is sufficiently small. This is achieved following successive deformations $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n, \dots$, of f_0 such that \tilde{f}_i^*h is approximately equal to $\tilde{f}_{i-1}^*h + \phi_i^2 d\psi_i^2$ for each i . Each step of deformation involves a convex integration (discussed in Section 3) and the deformation takes place on the open set U_λ containing $\text{supp } \phi_i$ in such a way that the value of the derivative $d\tilde{f}_{i-1}$ along $\tau_i = \ker d\psi_i$ is affected by a small amount. Since at most finitely many ϕ_i are non-zero on any U_λ , the sequence $\{\tilde{f}_i\}$ is eventually constant on each U_λ and therefore converges to a C^∞ map $\tilde{f} : M \rightarrow N$ which is very close to being a partial isometry. Indeed, for the final map \tilde{f} the total error $g_H - \tilde{f}^*h$, estimated by the function $n(g_H - \tilde{f}^*h)$, can be made arbitrarily small. Moreover, $d(f_0, \tilde{f})$ can be controlled by the function $n(g_H - f_0^*h)$. See Lemma 4.1 and Lemma 4.4 for a detailed proof.

Obtaining a partial isometry: The principal idea is to obtain a partial isometry as the limit of a sequence of C^∞ smooth g_H -short H -immersions $f_j : M \rightarrow (N, h)$ which is Cauchy in the fine C^1 -topology and is such that the induced metric f_j^*h approaches to g_H on H in the limit. More explicitly, the sequence $\{f_j\}$ will satisfy the following relations:

- (1) $n(g_H - f_j^*(h)) \approx \frac{1}{2}n(g_H - f_{j-1}^*(h))$,
- (2) $d(f_j, f_{j-1}) < c(m)n(g_H - f_{j-1}^*(h))^{\frac{1}{2}}$,

where $c(m)$ is a constant depending on the dimension m of the manifold M . The j -th map f_j can be seen as an improved approximate solution over f_{j-1} . The conditions (1) and (2) together guarantee that the sequence $\{f_n\}$ is a Cauchy sequence in the fine C^1 topology and hence it converges to some C^1 map $f : M \rightarrow N$. Then the induced metric f^*h must be equal to g_H when restricted to H by condition (1). Thus f is the desired partial isometry.

3. PRELIMINARIES OF CONVEX INTEGRATION THEORY

In this section, we recall from [5] and [10] the basic terminology of the theory of h -principle and preliminaries of convex integration theory.

Let M and N be smooth manifolds and $x \in M$. If $f : U \rightarrow N$ is a C^r map defined on an open subset U of M containing x , then the r -jet of f at x , denoted by $j_f^r(x)$, corresponds to the r -th degree Taylor's polynomial of f relative to a coordinate system around x . Let $J^r(M, N)$ denote the space of r -jets of germs of C^r -maps $M \rightarrow N$ and let $p^r : J^r(M, N) \rightarrow M$ be the natural projection map taking $j_f^r(x)$ to x , which is a fibration. For any C^r map $f : M \rightarrow N$, j_f^r is a section of p^r . Moreover, if $r > s$ then there is a canonical projection $p_s^r : J^r(M, N) \rightarrow J^s(M, N)$ which takes an r jet at x represented by a germ f to the s jet of f at x .

A *partial differential relation* of order r for C^r maps $M \rightarrow N$ is defined as a subspace \mathcal{R} of $J^r(M, N)$. If \mathcal{R} is an open subset then we say that \mathcal{R} is an *open* relation.

A section σ of $p^r : J^r(M, N) \rightarrow M$ is said to be a *section* of \mathcal{R} if the image of σ is contained in \mathcal{R} . A section of \mathcal{R} is often referred as a formal solution of \mathcal{R} . If $f : M \rightarrow N$ is such that j_f^r maps M into \mathcal{R} then f is called a *solution* of \mathcal{R} . A section $\sigma : M \rightarrow \mathcal{R}$ is called *holonomic* if $\sigma = j_f^r$ for a C^r -map $f : M \rightarrow N$.

Let $\Gamma(\mathcal{R})$ denote the space of sections of the r -jet bundle $p^r : J^r(M, N) \rightarrow M$ whose images lie in \mathcal{R} . We endow this space with the C^0 -compact open topology. The space of C^r solutions of \mathcal{R} is denoted by $\text{Sol } \mathcal{R}$. We endow it with the C^r compact-open topology. Then the r -th jet map $j^r : \text{Sol } \mathcal{R} \rightarrow \Gamma(\mathcal{R})$ defined by $j^r(f) = j_f^r$ is continuous relative to the given topologies.

Definition 3.1. *A relation \mathcal{R} is said to satisfy the h -principle if given a section σ of \mathcal{R} there exists a solution f of \mathcal{R} such that j_f^r is homotopic to σ in $\Gamma(\mathcal{R})$. If the r -jet map j^r is a weak homotopy equivalence then \mathcal{R} is said to satisfy the parametric h -principle.*

Let \mathcal{U} be a subspace of C^0 maps $M \rightarrow N$. A relation \mathcal{R} is said to satisfy the C^0 dense h -principle near \mathcal{U} provided given any $f \in \mathcal{U}$ and any neighbourhood N of $j_f^0(M)$, every section σ of \mathcal{R} which lies over j_f^0 (i.e., $p_0^r \circ \sigma = j_f^0$) admits a homotopy σ_t such that σ_t lies in $(p_0^r)^{-1}(\mathcal{U}) \cap \mathcal{R}$ and σ_1 is holonomic.

Given a differential relation \mathcal{R} , the main problem is to determine whether or not it has a solution. Proving h -principle is a step forward towards this goal. If a relation satisfies the h -principle then we can not at once say that the solution exists; however, we can conclude that if \mathcal{R} has a section (i.e., a formal solution) then it has a solution. Thus, we reduce a differential topological problem to a problem in algebraic topology. There are several techniques due to Gromov which address the question of h -principle. The convex integration theory is one such. Here we will review the convex integration theory only for first order differential relations.

Let τ be a codimension 1 integrable hyperplane distribution on M . Let f and g be germs at $x \in X$ of two C^1 smooth maps from M to N . We say that f and g are \perp -equivalent if

$$f(x) = g(x) \quad \text{and} \quad Df_x|_\tau = Dg_x|_\tau.$$

The \perp -equivalence is an equivalence relation on the 1-jet space $J^1(M, N)$. The equivalence class of $j_f^1(x)$ is denoted by $j_f^\perp(x)$ and is called the \perp -jet of f at x . Since τ is integrable, we can choose local coordinate systems $(U; x_1, \dots, x_{n-1}, t)$ so that $\{(x_1, \dots, x_{n-1}, t) : t = \text{const}\}$ are integral submanifolds of τ . Moreover, we can express $j_f^\perp(x)$ as $(j_f^\perp(x), \partial_t f(x))$, where $j_f^\perp = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n-1}})$ and $\partial_t f$ denotes the partial derivative of f in the direction of t . In particular, if $M = \mathbb{R}^n$, $N = \mathbb{R}^q$ and τ is defined by the codimension one foliation $\mathbb{R}^{n-1} \times \mathbb{R}$ on \mathbb{R}^n , then the 1-jet space gets a splitting $J^1(\mathbb{R}^n, \mathbb{R}^q) = J^\perp(\mathbb{R}^n, \mathbb{R}^q) \times \mathbb{R}^q$. The set of all \perp -jets, denoted by $J^\perp(M, N)$, has a manifold structure [10, 6.1.1] and the natural projection map $p_\perp^1 : J^1(M, N) \rightarrow J^\perp(M, N)$, taking a 1-jet to its \perp -equivalence class (relative to the given τ), defines an affine bundle, in which the fibres are affine spaces of dimension $n = \dim N$. The fibres of this affine bundle are called *principal subspaces* relative to τ . Note that there is a unique principal subspace through each point of $J^1(M, N)$. In fact, the fibre of $J^1(M, N) \rightarrow J^0(M, N)$ over any $b \in J^0(M, N)$ is foliated by these principal subspaces and the translation map takes principal subspaces onto principal subspaces.

Notation: We shall denote the principal subspace through $a \in J^1(M, N)$ by $R(a, \tau)$. If \mathcal{R} is a first order relation and $a \in \mathcal{R}$, then the connected component of a in $\mathcal{R} \cap R(a, \tau)$ will be denoted by $\mathcal{R}(a, \tau)$.

The following theorem, known as the h -Stability Theorem in the literature, ([5, 2.4.2(B)] and [10, Theorems 7.2, 7.17]) is a key result in the theory of convex integration.

Theorem 3.2. *Let \mathcal{R} be an open relation and let $f_0 : M \rightarrow N$ be a C^1 map such that*

- (1) $j_{f_0}^\perp$ lifts to a section σ_0 of \mathcal{R} and
- (2) $j_{f_0}^\perp(x)$ lies in the convex hull of $\mathcal{R}(\sigma_0(x), \tau_x)$ for every $x \in M$.

Let \mathcal{N} be any neighbourhood of $j_f^\perp(M)$. Then there exists a homotopy $\sigma_t : M \rightarrow \mathcal{R}$, $t \in [0, 1]$, such that

- (i) $\sigma_1 = j_{f_1}^\perp$ for some C^1 map $f_1 : M \rightarrow N$, so that f_1 is a solution of \mathcal{R} and
- (ii) $(p_1^\perp \circ \sigma_t)(M) \subset \mathcal{N}$ for all $t \in [0, 1]$. In particular f_1 is close to f_0 in the fine C^0 topology.

Further, if the initial map f_0 is a solution on $Op K$ for some closed set K then the homotopy remains constant on $Op K$.

Remark 3.3. *Since $C^\infty(M, N)$ is dense in $C^1(M, N)$ relative to the fine C^1 -topology and \mathcal{R} is open, we can perturb any C^1 -solution of \mathcal{R} to obtain a C^∞ solution.*

Definition 3.4. *A connected subset S in a vector space (or in an affine space) V is said to be ample if the convex hull of S is all of V . The subset defined by the polynomial $x^2 + y^2 - z^2 = 0$ in \mathbb{R}^3 is an example of an ample subset. However, the complement of a 2-dimensional vector subspace in \mathbb{R}^3 is not ample.*

A relation \mathcal{R} is said to be ample if for every hyperplane distribution τ on M , $\mathcal{R}(a, \tau)$ is ample in $R(a, \tau)$ for all $a \in \mathcal{R}$.

Theorem 3.5. ([5, 2.4.3, Theorem (A)]) *Every open ample relation satisfies the C^0 -dense parametric h -principle.*

We end this section with an application of Theorem 3.5 to the H -immersion relation; (see [4, 8.3.4] for an alternative proof).

Proposition 3.6. *Let M be a smooth manifold and H a subbundle of TM . Then H -immersions $f : M \rightarrow N$ satisfy the C^0 -dense parametric h -principle provided $\dim N > \text{rank } H$. In other words, every bundle map $(F_0, f_0) : TM \rightarrow TN$ such that $F_0|_H$ is a monomorphism is homotopic through such bundle maps to an $(F, f) : TM \rightarrow TN$ such that $F = df$ provided $\dim N > \text{rank } H$.*

Proof. The H -immersions $f : M \rightarrow N$ are solutions to the first order partial differential relation

$$\mathcal{R} = \{(x, y, \alpha) \in J^1(M, N) \mid \alpha|_{H_x} : H_x \rightarrow T_y N \text{ is injective linear}\}.$$

First of all, we prove that \mathcal{R} is an open relation. Recall that if (U, ϕ) and (V, ψ) are coordinate charts in M and N respectively then the bijection $\tau : J^1(U, V) \rightarrow J^1(\phi(U), \psi(V)) = \phi(U) \times \psi(V) \times L(\mathbb{R}^m, \mathbb{R}^n)$ defined by

$$\tau(j_f^\perp(x)) = (\phi(x), \psi(f(x)), d(\psi f \phi^{-1})_{\phi(x)})$$

is a coordinate chart for the total space $J^1(M, N)$ of the 1-jet bundle [7], where $m = \dim M$ and $n = \dim N$. Since H is a subbundle of TM we can further choose a trivialisation $\Phi : TM|_U \rightarrow U \times \mathbb{R}^m$ of the bundle $TM|_U$ (possibly after shrinking U), such that Φ maps H onto $U \times \mathbb{R}^k$. Then $\bar{\tau} : J^1(U, V) \rightarrow \phi(U) \times \psi(V) \times L(\mathbb{R}^m, \mathbb{R}^n)$ by $\bar{\tau}(j_f^1(x)) = (\phi(x), \psi(f(x)), d(\psi f)_x \circ \bar{\Phi}_{\phi(x)}^{-1})$ is a diffeomorphism, where $\bar{\Phi} = (\phi \times Id) \circ \Phi : TM|_U \rightarrow \phi(U) \times \mathbb{R}^m$.

Now, consider the restriction morphism $r : L(\mathbb{R}^m, \mathbb{R}^n) \rightarrow L(\mathbb{R}^k, \mathbb{R}^n)$ which takes a linear transformation $L \in L(\mathbb{R}^m, \mathbb{R}^n)$ onto its restriction $L|_{\mathbb{R}^k}$. Let $L_k(\mathbb{R}^m, \mathbb{R}^n)$ denote the inverse image under r of the set of all monomorphisms $\mathbb{R}^k \rightarrow \mathbb{R}^n$. This is clearly an open set and it is easy to see that $\bar{\tau}$ maps $\mathcal{R} \cap J^1(U, V)$ diffeomorphically onto $\phi(U) \times \psi(V) \times L_k(\mathbb{R}^m, \mathbb{R}^n)$. Consequently \mathcal{R} is open in the 1-jet space $J^1(M, N)$.

Next, we shall show that \mathcal{R} is an ample relation. To see this, consider a codimension 1 subspace τ_x of TM_x for some $x \in M$ and take a 1-jet $j_f^1(x) \in \mathcal{R}$. We need to show that the principal subspace

$$R(j_f^1(x), \tau_x) = \{(x, f(x), \beta) \in J^1(M, N) | \beta = df_x \text{ on } \tau_x\}$$

intersects the relation \mathcal{R} in a pathconnected set and moreover the convex hull of the intersection, denoted by $\mathcal{R}(j_f^1(x), \tau_x)$, is all of $R(j_f^1(x), \tau_x)$. There are two possible cases:

Case 1. $H_x \subset \tau_x$. In this case, the principal subspace is completely contained in \mathcal{R} . Thus $\mathcal{R}(j_f^1(x), \tau_x)$ is equal to the principal subspace itself.

Case 2. $H_x \cap \tau_x$ is a codimension 1 subspace of H_x . Choose a vector $v \in H_x$ which is transverse to $H_x \cap \tau_x$. First observe that $R(j_f^1(x), \tau_x)$ is affine isomorphic to $T_{f(x)}N$ since any 1-jet (x, y, β) is completely determined by $\beta(v)$. Therefore, $\mathcal{R}(j_f^1(x), \tau_x)$ is affine equivalent to the subset

$$S(j_f^1(x)) = \{w \in T_{f(x)}N | w \notin df_x(\tau_x \cap H_x)\}.$$

Since $\tau_x \cap H_x$ has dimension $k-1$ and df_x is injective on H_x , the subspace $df_x(\tau_x \cap H_x)$ is of codimension at least 2 in $T_{f(x)}N$ provided $\dim N > k$. Hence $S(j_f^1(x))$ is path-connected and its convex hull is all of $T_{f(x)}N$. In other words, the convex hull of $\mathcal{R}(j_f^1(x), \tau_x)$ is all of $R(j_f^1(x), \tau_x)$.

This proves that \mathcal{R} is an open, ample relation. Hence, we can apply Theorem 3.5 to conclude that \mathcal{R} satisfies the C^0 -dense parametric h principle. \square

Corollary 3.7. *Suppose that $f_0 : M \rightarrow N$ is a smooth map. If $\dim N \geq \dim M + \text{rank } H$, then f_0 can be homotoped within its C^0 -neighbourhood to a C^∞ H -immersion $f : M \rightarrow N$.*

Proof. In view of the above proposition it is enough to show that f_0 can be covered by a monomorphism $F : H \rightarrow TN$ if $\dim N \geq \dim M + \text{rank } H$. It is well-known that the obstruction to the existence of such an F lies in certain homotopy groups of the Stiefel manifold $V_k(\mathbb{R}^n)$, namely in $\pi_i(V_k(\mathbb{R}^n))$ for $0 \leq i \leq m-1$, where $m = \dim M$, $n = \dim N$ and $k = \text{rank } H$. Since $V_k(\mathbb{R}^n)$ is $n-k-1$ connected the obstructions vanish for $n \geq m+k$. This proves the corollary. \square

Remark 3.8. *The set of smooth H -immersions $M \rightarrow N$ is an open, dense subset of $C^\infty(M, N)$ relative to the fine C^∞ topology when $\dim N \geq \dim M + \text{rank } H$ [2, Proposition 2.2].*

4. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

Let (N, h) be a smooth Riemannian manifold of dimension n and (M, H, g_H) be as in Section 1. Let g_0 be a Riemannian metric on M such that $g_0|_H = g_H$. Suppose that $\dim N > \text{rank } H$.

Lemma 4.1. (*Main Lemma*) *Let g be a Riemannian metric on H such that $g < g_H$. Suppose that $f : M \rightarrow N$ is a smooth H -immersion and $g - f^*h = \phi^2 d\psi^2$ on H , where ϕ and ψ are smooth real valued functions on M such that $\text{supp } \phi$ is contained in a coordinate neighbourhood U and $d\psi$ has rank 1 on U . Given any two positive functions ε and δ on M , f can be homotoped to a C^∞ map $\tilde{f} : M \rightarrow N$ in a given C^0 neighbourhood of f such that \tilde{f} coincides with f outside U and satisfies the following properties:*

- (i) \tilde{f} is an H -immersion,
- (ii) $0 \leq n_{g_H}(g - \tilde{f}^*h) < \delta$,
- (iii) $d_{g_0}(f, \tilde{f}) < n_{g_H}(g - f^*h)^{\frac{1}{2}} + \varepsilon$.

Remark 4.2. *Observe that inequality (iii) in the lemma above is independent of any particular choice of g_0 .*

Proof. We will prove the lemma by an application of Theorem 3.2. First observe that the partial isometries $(M, H, g) \rightarrow (N, h)$ are solutions to a first order differential relation \mathcal{I} given by:

$$\mathcal{I} = \{(x, y, \alpha) \in J^1(M, N) \mid \alpha : T_x M \rightarrow T_y N \text{ is linear and } \alpha^*h = g \text{ on } H_x\}.$$

Let f be as in the hypothesis: $g - f^*h = \phi^2 d\psi^2$ on H , where $d\psi$ is of rank 1 on U . Then the kernel of $d\psi$ defines a codimension 1 integrable distribution on U which we will denote by τ . We construct \perp -jet bundle $J^\perp(M, N) \rightarrow M$ relative to this hyperplane distribution τ as described in Section 3. Recall that two 1-jets (x, y, α) and (x, y, β) in $J^1(M, N)$ are equivalent if $\alpha|_{\tau_x} = \beta|_{\tau_x}$, and the \perp -jets are equivalence classes of 1-jets under the above equivalence relation. The set of all 1-jets equivalent to (x, y, α) is an affine subspace of $J^1_{(x, y)}(M, N)$. This is referred as the principal subspace containing α and is denoted by $R(\alpha, \tau)$. We claim that for all $x \in U$

- (a) $R(j_f^1(x), \tau_x) \cap \mathcal{I}$ is a non-empty path-connected set.
- (b) $j_f^1(x)$ belongs to the convex hull of $R(j_f^1(x), \tau_x) \cap \mathcal{I}$.
- (c) There is a section σ_0 of \mathcal{I} such that $\sigma_0(x) \in R((j_f^1(x), \tau_x))$.

Let V denote the open subset of U consisting of all points x such that $g - f^*h|_{H_x} \neq 0$. We shall first prove the statements (a), (b) and (c) for points in V . If $x \in V$ then H_x is not contained in τ_x because $g - f^*h = \phi^2 d\psi^2$ on H_x , and therefore $H_x \cap \tau_x$ is of codimension 1 in H_x . Choose a smooth unit vector field \mathbf{v} on the open set V such that $\mathbf{v}_x \in H_x$ and is g -orthogonal to $H_x \cap \tau_x$ for all $x \in V$. Observe that every $\beta \in R(j_f^1(x), \tau_x)$ is then completely determined by its value on the vector \mathbf{v}_x . In fact, we can define an affine isomorphism $R(j_f^1(x), \tau_x) \rightarrow T_y N$ by $\beta \mapsto \beta(\mathbf{v}_x)$ which maps $\mathcal{I} \cap R(j_f^1(x), \tau_x)$ onto the set

$$S_x = \{w \in T_{f(x)}N \mid w \perp_h df_x(H_x \cap \tau_x), \|w\|_h = 1\}.$$

Denote the rank of H by k . Since $df_x|_{H_x}$ is injective linear, S_x represents the unit sphere in a codimension $(k-1)$ subspace of $T_y N$. Hence, for $n > k$, S_x is path-connected. This proves (a).

Also, $df_x(\mathbf{v}_x)$ lies in the convex hull of S_x . Indeed, the condition $g - f^*h = \phi^2 d\psi^2$ on H_x implies that $h(df_x(\mathbf{v}_x), df_x(w)) = 0$ for all $w \in H_x \cap \tau_x$ and therefore, $df_x(\mathbf{v}_x)$ is h -orthogonal to $df_x(H_x \cap \tau_x)$. Moreover, as $g - f^*h|_{H_x} > 0$ and f is an H -immersion it also follows that $0 < \|df_x(\mathbf{v}_x)\| < 1$. Hence, $df_x(\mathbf{v}_x)$ lies in the convex hull of S_x proving (b).

To prove (c) note that $df_x(\mathbf{v}_x)$ is orthogonal to $df_x(H_x \cap \tau_x)$. Therefore, if we define $w_0(x) = df_x(\mathbf{v}_x) / \|df_x(\mathbf{v}_x)\|_h$ then $w_0(x) \in S_x$. Let $\sigma_0(x)$ denote the 1-jet in $R(j_f^1(x), \tau_x) \cap \mathcal{I}$ which corresponds to $w_0(x)$. Thus, σ_0 is a continuous section of \mathcal{I} over V as mentioned in (c).

If $x \in U \setminus V$, then either $\phi(x) = 0$ or $d\psi_x|_{H_x} = 0$ i.e., H_x is contained in $\tau_x = \ker d\psi_x$. If $\phi(x) = 0$ then proceeding as in the above case we can prove that (a) and (b) are true. If $H_x \subset \tau_x$ then the principal subspace $R(j_f^1(x), \tau_x)$ is completely contained in \mathcal{I} . Therefore (a) and (b) are clearly true in this case also. Further, $x \in U \setminus V$ implies that $j_f^1(x) \in \mathcal{I}$ and we can choose $\sigma_0(x) = j_f^1(x)$ on $U \setminus V$ so that (c) is proved on all of U . This completes the proof of the claim made above.

In fact, we have proved that the map $f : M \rightarrow N$ satisfies both (1) and (2) of the hypothesis of Theorem 3.2 relative to the relation \mathcal{I} . Indeed, by our construction σ_0 lifts j_f^1 . Further, $j_f^1(x)$ lies in the convex hull of $\mathcal{I}(\sigma_0(x), \tau_x)$ for all $x \in U$. This follows from (a) and (b) since $R(j_f^1(x), \tau_x) \cap \mathcal{I} = R(\sigma_0(x), \tau_x) \cap \mathcal{I} = \mathcal{I}(\sigma_0(x), \tau_x)$ (see section 3). However, we cannot apply Theorem 3.2 to (f, \mathcal{I}) , since \mathcal{I} is not open. To surpass this difficulty, we consider a small open neighbourhood $Op\mathcal{I}$ of \mathcal{I} in the H -immersion relation \mathcal{R} and apply Theorem 3.2 to the pair $(f, Op\mathcal{I})$ to obtain a smooth H -immersion $\tilde{f} : M \rightarrow N$ which is a solution of $Op\mathcal{I}$. By choosing $Op\mathcal{I}$ sufficiently small we can ensure that $\tilde{f}^*h|_H$ is arbitrarily close to g . Thus, we prove (i) and (ii) as stated in the theorem.

In order that \tilde{f} satisfies condition (iii) as well, we need to modify the relation $Op\mathcal{I}$ further. Consider the subset $S'_x = \{w \in S_x | h(w, df_x(\mathbf{v}_x)) \geq h(df_x(\mathbf{v}_x), df_x(\mathbf{v}_x))\}$ of S_x (see [4, §21.5]). This is path-connected, symmetric about $w_0(x)$ and contains $df_x(\mathbf{v}_x)$ in its convex hull. Moreover, for any vector w in S'_x , $\|w - df_x(\mathbf{v}_x)\| \leq \sqrt{1 - \|df_x(\mathbf{v}_x)\|_h^2}$. Let \mathcal{I}' denote the subset of \mathcal{I} defined by S'_x , $x \in M$. Now, applying Theorem 3.2 to $(Op\mathcal{I}', f)$ we obtain a C^∞ map $\tilde{f} : M \rightarrow N$ which is homotopic to f and is a solution of $Op\mathcal{I}'$. As we have already observed, \tilde{f} satisfies (i) and (ii) as stated in the theorem. Further, we have,

$$\begin{aligned} \|d\tilde{f}_x(\mathbf{v}_x) - df_x(\mathbf{v}_x)\|_h &\leq \sqrt{1 - \|df_x(\mathbf{v}_x)\|_h^2} + \varepsilon \\ &= \sqrt{(g - f^*h)(\mathbf{v}_x, \mathbf{v}_x)} + \varepsilon \end{aligned}$$

where the ‘error term’ ε appears because of enlarging \mathcal{I} . Since $g_0|_H = g_H$ and $\mathbf{v}_x \in H$, dividing out both sides by $\|\mathbf{v}_x\|_{g_0}$ we obtain from the above that

$$\frac{\|d\tilde{f}_x(\mathbf{v}_x) - df_x(\mathbf{v}_x)\|_h}{\|\mathbf{v}_x\|_{g_0}} \leq \sqrt{n_{g_H}(g - f^*h)} + \varepsilon.$$

Moreover, by Theorem 3.2 we can choose \tilde{f} so that the directional derivatives of \tilde{f} along τ are arbitrarily close to the corresponding derivatives of f . Thus we obtain that $d_{g_0}(f, \tilde{f}) \leq \sqrt{n_{g_H}(g - f^*h)} + \varepsilon$. \square

Remark 4.3. In the above lemma we started with a C^∞ map f satisfying $f^*h < g < g_H$ and given any $\delta > 0$ obtained an \tilde{f} satisfying the condition $n(g - \tilde{f}^*h) < \delta$. Therefore, if we choose δ sufficiently small then \tilde{f} can be made to satisfy the inequality $f^*h < \tilde{f}^*h < g_H$.

We now fix a countable open covering $\mathcal{U} = \{U_\lambda | \lambda \in \Lambda\}$ of the manifold M which has the following properties:

- (a) each U_λ is a coordinate neighbourhood in M and
- (b) for any λ_0 , U_{λ_0} intersects atmost $c_1(m)$ many U_λ 's including itself,

where $c_1(m)$ is an integer depending on $m = \dim M$. This open covering will remain fixed throughtout. All decompositions of Riemannian metrics on H will be considered with respect to this covering.

Lemma 4.4. Let $f_0 : M \rightarrow N$ be a smooth H -immersion such that $f_0^*h < g_H$ on H . Then f_0 can be homotoped to a C^∞ map f_1 in any given C^0 neighbourhood of f_0 such that

- (i) f_1 is an H -immersion and $f_1^*h < g_H$ on H ,
- (ii) $0 < n_{g_H}(g_H - f_1^*h) < \frac{2}{3}n_{g_H}(g_H - f_0^*h)$,
- (iii) $d_{g_0}(f_0, f_1) < c(m)\sqrt{n_{g_H}(g_H - f_0^*h)}$,

where $c(m)$ is a constant which depends on the dimension m of M .

Proof. Since $g_H - f_0^*h|_H > 0$ we get a decomposition as follows:

$$g_H - f_0^*h = 2 \sum_{k=1}^{\infty} \phi_k^2 d\psi_k^2 \text{ on } H,$$

where ϕ_k and ψ_k are as described in Lemma 2.2. It further follows from the lemma that all but finitely many ϕ_i vanish on any U_p and at most $c(m)$ number of ϕ_i are non-vanishing at any point x . Define a sequence of Riemannian metrics on H as follows: $\bar{g}_0 = f_0^*h|_H$ and $\bar{g}_k = \bar{g}_{k-1} + \phi_k^2 d\psi_k^2|_H$. Then each $\bar{g}_k < g_H$ and $\lim_{k \rightarrow \infty} \bar{g}_k = f_0^*h + \frac{1}{2}(g - f_0^*h)$. In successive steps we aim to increment the induced metric on H by $\phi_k^2 d\psi_k^2|_H$, $k = 1, 2, \dots$. However, in the process of achieving this we admit an error in each step; the error in step k is denoted by δ_k . Thus at the end of the k -th step we will have a map \bar{f}_k such that $\bar{f}_k^*h|_H = \bar{g}_k + \sum_{i=1}^k \delta_i$, $k = 1, 2, \dots$. Explicitly, we will construct a sequence of smooth maps $\{\bar{f}_k\}$, $k = 1, 2, \dots$, such that $\bar{f}_k^*h|_H = \bar{g}_k + \delta_k$, $k = 1, 2, \dots$ which satisfy the following conditions:

- (1) \bar{f}_k is a g_H -short H -immersion, and $\bar{f}_k = \bar{f}_{k-1}$ outside U_k ,
- (2) $0 \leq n_{g_H}(\bar{\delta}_k - \bar{\delta}_{k-1}) < \delta'_k$,
- (3) $d_{g_0}(\bar{f}_{k-1}, \bar{f}_k) < n_{g_H}(\bar{g}_k - \bar{g}_{k-1})^{1/2} + \varepsilon_k$,
- (4) $\bar{g}_{k+1} + \bar{\delta}_k < g_H$.

where $\bar{f}_0 = f_0$ and $\delta_0 = 0$ and $\sum_{k \geq 1} \varepsilon_k < \infty$,

Taking $g = \bar{g}_1$ and $f = \bar{f}_0$ in Lemma 4.1 we can prove the first step of the induction for $k = 1$. Let $\bar{f}_1^*h = \bar{g}_1 + \delta_1$, where δ_1 is such that $\bar{g}_2 + \delta_1 < g_H$. Suppose we have obtained \bar{f}_k satisfying (1)–(4) at the end of the k -th step, where we can write $\bar{f}_k^*h|_H = \bar{g}_k + \delta_k$. In the next step we want to increment the induced metric on H by a quantity $\phi_{k+1}^2 d\psi_{k+1}^2|_H$, that is, we want to induce $\bar{g}_{k+1} + \bar{\delta}_k$ on H . Taking $g = \bar{g}_{k+1} + \bar{\delta}_k$ and $f = \bar{f}_k$ in Lemma 4.1 we obtain a smooth map \bar{f}_{k+1} which clearly

satisfies (1). If we write $\bar{f}_{k+1}^* h = \bar{g}_{k+1} + \bar{\delta}_k + \delta_{k+1}$, then $\bar{\delta}_{k+1} = \sum_{i=1}^{k+1} \delta_i$ and hence (2) and (3) are clearly satisfied. Finally, for sufficiently small choice of δ'_{k+1} , we can make δ_{k+1} satisfy $\bar{g}_{k+2} + \bar{\delta}_{k+1} < g_H$. This completes the construction of $\{\bar{f}_k\}$ by induction.

Let $f_1 = \lim_{k \rightarrow \infty} \bar{f}_k$. We want to show that the sequence $\{\bar{f}_k\}$ is eventually constant on any U_λ , which will imply that f_1 is a C^∞ map. To see this take a fixed $p \in \Lambda$. It follows from Lemma 2.2 that the set of integers

$$\{i \mid \text{supp } \phi_i \subset U_{\lambda'} \text{ where } U_{\lambda'} \cap U_\lambda \neq \emptyset\}$$

is finite. Let i_λ denote the maximum element of this set. Observe that if $i > i_\lambda$ and $\text{supp } \phi_i \subset U_{\lambda'}$ then $U_\lambda \cap U_{\lambda'} = \emptyset$, so that ϕ_i vanishes identically on U_λ . This implies that $\bar{f}_i = \bar{f}_{i-1}$ on U_λ by the given construction. Thus, f_1 is a smooth map and $f_1^* h = f_0^* h + \frac{1}{2}(g - f_0^* h) + \sum_k \delta_k$. We shall prove that f_1 is the desired map. To see this note that

$$\begin{aligned} n_{g_H}(g_H - f_1^* h) &= n_{g_H}(g_H - [f_0^* h + \frac{1}{2}(g_H - f_0^* h) + \sum_k \delta_k]) \\ &= n_{g_H}(\frac{1}{2}(g_H - f_0^* h) - \sum_k \delta_k) \\ &\leq \frac{1}{2}n_{g_H}(g_H - f_0^* h) + \sum_k n_{g_H}(\delta_k) \text{ (since the sum is locally finite)} \\ &\leq \frac{1}{2}n_{g_H}(g_H - f_0^* h) + \sum_k \delta'_k \end{aligned}$$

We can choose δ'_k at each stage so that $\sum_k \delta'_k < \frac{1}{6}n_{g_H}(g_H - f_0^* h)$, and we obtain relation (ii). On the other hand,

$$\begin{aligned} d_{g_0}(f_0, f_1) &\leq \sum_{k \geq 1} d_{g_0}(\bar{f}_{k-1}, \bar{f}_k) \\ &\leq \sum_{k \geq 1} n_{g_H}(\bar{g}_k - \bar{g}_{k-1})^{1/2} + \sum_{k \geq 1} \varepsilon_k \text{ by (3) above} \end{aligned}$$

Each term of the first series on the right hand side can be estimated as follows:

$$\begin{aligned} n_{g_H}(\bar{g}_{k+1} - \bar{g}_k)^{1/2} &\leq n_{g_H}(g_H - \bar{g}_k)^{1/2} \\ &= n_{g_H}(g_H - \bar{f}_0^* h)^{1/2}. \end{aligned}$$

However, since $\bar{g}_k - \bar{g}_{k-1} = \phi_k^2 d\psi_k^2$ and at most $c(m)$ number of ϕ_k are non-vanishing at a point, the series $\sum_{k \geq 1} n_{g_H}(\bar{g}_k - \bar{g}_{k-1})^{1/2}$ is bounded above by $c(m)n_{g_H}(g_H - \bar{f}_0^* h)^{1/2}$. On the other hand, by Lemma 4.1 we are allowed to choose the sequence $\{\varepsilon_k\}$ so that $\sum_k \varepsilon_k < \infty$. This gives the desired relation (iii). \square

We have made all necessary preparation for the proof of Theorem 1.1.

Proof. of Theorem 1.1. Let f_0 be as in the hypothesis of the theorem. Applying Lemma 4.4 on f_0 recursively we can construct a sequence of C^∞ maps $\{f_i : M \rightarrow N : i = 1, 2, \dots\}$ which has the following properties.

- (1) $0 < f_i^* h < g_H$ on H ,
- (2) $0 < n(g_H - f_i^* h) < \frac{2}{3}n(g_H - f_{i-1}^* h)$,

$$(3) \ d(f_{i-1}, f_i) < c(m)n(g_H - f_{i-1}^*h)^{1/2} \text{ for all } i = 1, 2, \dots$$

It follows from (2) that the sequence of metrics $f_i^*h|_H$, $i = 1, 2, \dots$, converges to g_H and therefore, $\{f_i\}$ is Cauchy in the fine C^1 topology by (3). Hence, $\{f_i\}$ must converge to some C^1 map $f : M \rightarrow N$. Consequently, $\lim_{i \rightarrow \infty} f_i^*h|_H = f^*h|_H$. Thus $f^*h|_H = g_H$ and f is the desired partial isometry. The homotopy between f and f_0 is obtained by concatenating the homotopies between f_i and f_{i+1} for $i = 1, 2, \dots$. \square

Remark 4.5. *We have proved something stronger than what we claimed in Theorem 1.1. We have proved that the partial isometries satisfy the C^0 -dense h -principle in the space of g_H -short H -immersions.*

Proof. of Corollary 1.2. In view of Theorem 1.1 it is enough to obtain a g_H short H -immersion $f : M \rightarrow \mathbb{R}^n$ for $n \geq m + k$, where $m = \dim M$ and $k = \text{rank } H$. We first observe that such a map f exists if n is sufficiently large (see [9]). If $n \leq m + k$ then we have nothing more to show. If $n > m + k$, then we will show that there exists a vector $v \in \mathbb{R}^n$ such that $P_v \circ f$ is an H -immersion, where P_v denotes the orthogonal projection of \mathbb{R}^n onto v^\perp .

We first cover M by countably many open neighbourhoods U_j such that $TM|_{U_j}$ is trivial. We may assume that under the trivialising map $H|_{U_j}$ sits inside $U_j \times \mathbb{R}^n$ as $U_j \times \mathbb{R}^k$. A vector $v \in \mathbb{R}^n$ for which $P_v \circ f$ is not an H -immersion on U_j corresponds to a pair $(x, u) \in U_j \times \mathbb{R}^k$ such that $df_x(u)$ is a scalar multiple of v . Thus, for $v \in S^{n-1}$, $P_v \circ f$ is not an H -immersion on U_j if and only if v lies in the image of the map $F : U_j \times S^{k-1} \rightarrow S^{n-1}$ given by $(x, u) \mapsto \frac{df_x(u)}{\|df_x(u)\|}$. If $n > m + k$ then the image of this map is a set of measure zero by Sard's theorem [7]. Since M can be covered by countably many U_j 's and the countable union of sets of measure zero is again a set of measure zero, we have proved that $P_v \circ f$ is an H -immersion for almost all $v \in \mathbb{R}^n$. Finally, we observe that the projection operators are length decreasing. Hence, $P_v \circ f$ is also a g_H short H -immersion since f is so. Hence M admits a g_H -short H -immersion $(M, g_H) \rightarrow (\mathbb{R}^n, g_{can})$ for $n \geq \dim M + \text{rank } H$. \square

In the special situation, when H is an integrable subbundle, we can reformulate Corollary 1.2 as follows.

Corollary 4.6. *Every Riemannian manifold (M, g_0) with a regular foliation \mathcal{F} admits a C^1 -map $f : M \rightarrow \mathbb{R}^n$ which restricts to an isometric immersion on each leaf of the foliation, provided $n \geq \dim M + \dim \mathcal{F}$.*

Remark 4.7. *We observed in Section 1 that a partial isometry of a sub-Riemannian manifold (M, H, g_H) is also a path isometry with respect to the Carnot-Caratheodory metric d_H on M induced by g_H . Therefore, by Corollary 1.2 there is a path-isometry $f : (M, d_H) \rightarrow (\mathbb{R}^n, d_{can})$, provided $n \geq \dim M + \text{rank } H$. We refer to a result in [3, Corollary 1.5] which is of similar interest.*

5. APPLICATIONS OF THEOREM 1.1

In this section we discuss some applications of Theorem 1.1. Throughout, we assume M to be a closed manifold. First observe that if M is a closed manifold and N is an Euclidean space, then the hypothesis of Theorem 1.1 can be relaxed

to conclude the existence of partial isometry. Indeed, we do not require the g_H -shortness condition on f_0 ; given any H -immersion $f_0 : M \rightarrow \mathbb{R}^n$ we can obtain a g_H -short H -immersion f_1 which is of the form λf_0 , where λ is a positive real number. Applying Theorem 1.1 we can then homotope f_1 to a partial isometry $f : M \rightarrow \mathbb{R}^n$. However, the resulting partial isometry cannot be made C^0 -close to f_0 by this technique, since $f_1 = \lambda f_0$ may not be C^0 -close to f_0 .

Corollary 5.1. ([5]) *Let M be a closed manifold and ∂_i , $i = 1, 2, \dots, k$, be linearly independent vector fields on M . Then there exists a C^1 map $f : M \rightarrow \mathbb{R}^{k+1}$ such that $\langle \partial_i f, \partial_j f \rangle = \delta_{ij}$, $1 \leq i \leq j \leq k$, where $\delta_{ij} = 1$ if $i = j$ and 0 if $i < j$.*

Proof. Let H be the (trivial) subbundle of TM spanned by the vector fields ∂_i , $i = 1, 2, \dots, k$. Define a Riemannian metric g_H on H by the relations

$$g_H(\partial_i, \partial_j) = 0 \text{ if } i \neq j \text{ and } g_H(\partial_i, \partial_i) = 1$$

for $i, j = 1, \dots, k$. Consider the triple (M, H, g_H) as defined above. Since H is trivial, Proposition 3.6 guarantees the existence of an H -immersion $M \rightarrow \mathbb{R}^{k+1}$ which can be scaled appropriately in order to get a strictly g_H short H -immersion, since the manifold M is closed. Hence by Theorem 1.1 there exists a C^1 partial isometry $f : (M, g_H) \rightarrow (\mathbb{R}^{k+1}, g_{can})$. This means that $\langle \partial_i f, \partial_j f \rangle = g_H \langle \partial_i, \partial_j \rangle$ and the proof is now complete. \square

Remark 5.2. *A more general form of the above result is in fact true. Let $\Sigma(k, \mathbb{R})$ denote the set of all positive definite symmetric matrices over reals and let $g : M \rightarrow \Sigma(k, \mathbb{R})$ be any smooth map. Then g can be realised as the matrix $(\langle \partial_i f, \partial_j f \rangle)_{i,j}$ for some C^1 -function $f : M \rightarrow \mathbb{R}^{k+1}$ provided M is a closed manifold.*

Gromov observed in [5] that if we have $k = 1$ in Corollary 5.1 then we can actually obtain C^∞ partial isometries. We here give a direct proof of this result without going into the convex integration theory.

Theorem 5.3. *If M is a closed manifold and X is a smooth nowhere vanishing vector field on M , then there exists a C^∞ -map $f : M \rightarrow \mathbb{R}^2$ such that $\langle Xf, Xf \rangle = 1$.*

Proof. of Theorem 5.3 Let X be a smooth vector field on M which is nowhere vanishing. We need to solve the equation $\langle Xf, Xf \rangle = 1$, for smooth functions $f : M \rightarrow \mathbb{R}^2$. Let H denote the 1-dimensional (integrable) distribution on M determined by X . By Proposition 3.6 there exists an H -immersion $f_0 : M \rightarrow \mathbb{R}^2$ which implies that Xf_0 is a nowhere vanishing function on M . Since M is a closed manifold, without loss of generality we may assume that $0 < \langle Xf_0, Xf_0 \rangle = \phi^2 < 1$. This condition means that f is g_H -short if we define g_H by $g_H(X, X) = 1$. Consider the equation $\langle X(f_0 + \alpha), X(f_0 + \alpha) \rangle = 1$, where $\alpha : M \rightarrow \mathbb{R}^2$ is a smooth map. This reduces to

$$(1) \quad \langle X\alpha, X\alpha \rangle + 2\langle Xf_0, X\alpha \rangle = 1 - \phi^2.$$

We split this into a system of two equations as follows:

$$(2) \quad \langle Xf_0, X\alpha \rangle = 0 \quad \langle X\alpha, X\alpha \rangle = 1 - \phi^2.$$

Now note that $\beta = \frac{\sqrt{1-\phi^2}}{\|Xf_0\|} \rho_{\pi/2} \circ (Xf_0)$ is a formal solution of the above system, where $\rho_{\pi/2}$ is the rotation on \mathbb{R}^2 through the angle $\pi/2$ in the anticlockwise direction. Hence the problem of finding the desired f reduces to solving the equation $X\alpha = \beta$, where $\beta : M \rightarrow \mathbb{R}^2$ is a nowhere vanishing smooth function.

The vector field X can be considered as a first order linear differential operator on $C^\infty(M, \mathbb{R}^2)$. We will define a differential operator $M : C^\infty(M, \mathbb{R}^2) \rightarrow C^\infty(M, \mathbb{R}^2)$ such that $X(M(\beta)) = \beta$ for all $\beta \in C^\infty(M, \mathbb{R}^2)$. We first observe that this problem is local and therefore it is enough to define local inversion operators on open sets around each point [5, 2.3.8]. To see this let $\{U_\mu\}$ be a locally finite open covering of M by coordinated neighbourhoods on each of which we have a local inversion M_μ of X . Define M by $M\beta = \sum_\mu M_\mu(\phi_\mu\beta)$ where $\{\phi_\mu\}$ is a partition of unity subordinate to the open covering $\{U_\mu\}$. Then M is a global inversion operator since

$$\begin{aligned} X(M\beta) &= \sum_\mu X(M_\mu(\phi_\mu\beta)) \quad (\text{since the sum is locally finite}) \\ &= \sum_\mu \phi_\mu\beta \quad (\text{since } \phi_\mu\beta \text{ is supported on } U_\mu) \\ &= \beta \end{aligned}$$

It now remains to prove the local existence of an inversion M of X . Recall that the distribution H is integrable so that M is foliated by integral curves of H . Indeed around each point of M there exists a coordinate system $(U, (x, t))$ such that U is homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$ and $\frac{\partial}{\partial t}$ is tangent to H , so that X can be expressed as $\psi(x, t)\frac{\partial}{\partial t}$ on U . Therefore, the problem reduces to solving the equation $\psi(x, t)\frac{\partial\alpha}{\partial t} = \beta(x, t)$. Since $\psi(x, t)$ is nowhere vanishing we can define $M\beta$ by $\int_0^t \beta(x, t)/\psi(x, t) dt$. This completes the proof. \square

Remark 5.4. *Possibly we do not require the closedness condition on M in the above two results (see [5]).*

We end this section with an example of C^∞ partial isometry which supports the above theorem.

Example 5.5. Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the smooth immersion defined by

$$\psi(\theta, \phi) = ((b + a \cos \theta) \cos \phi, (b + a \cos \theta) \sin \phi, a \sin \theta),$$

where $(\theta, \phi) \in \mathbb{R}^2$ and a, b are two real numbers with $0 < a < b$. The image of ψ is a 2-torus \mathbb{T}^2 which is a manifold with local parametrisations defined by ψ . Let g denote the flat metric on \mathbb{T}^2 induced by ψ .

For any real number α , we have a 1-dimensional foliation of $\mathbb{R} \times \mathbb{R}$ by lines of slope α . Let \mathcal{F}_α denote the image of this foliation on \mathbb{T}^2 under ψ , and H_α the corresponding 1-dimensional distribution on \mathbb{T}^2 . Let g_α be the restriction of g to H_α . Consider the map $f : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y, z) = (x, y)$. It is easy to see that f is g_α -short H_α -immersion. In particular, if $\alpha = 0$ then f_0 itself is a partial isometry.

REFERENCES

- [1] G. D'Ambra, M. Datta: Isometric C^1 -immersions for pairs of Riemannian metrics. *Asian J. Math.* 6 (2002), no. 2, 373–384.
- [2] G. D'Ambra, R. de Leo, A. Loi: Partially isometric immersion and free maps, arXiv:1007.3024v1 [math.DG] 18 Jul 2010.

- [3] E. Le Donne: Lipschitz and path-isometric embeddings of metric spaces, arXiv:1005.1623v1 [math.MG] 10 May 2010.
- [4] Y. Eliashberg and N. Mishachev: *Introduction to the h-Principle* Graduate Studies in Mathematics, **48**, American Mathematical Society.
- [5] M. Gromov: *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge Band 9. Springer-Verlag, 1986.
- [6] A. Bellaïche, J. Risler (Eds.): *Sub-Riemannian geometry*, Progr. Math., 144, 1996.
- [7] M. Golubitsky, V. Guillemin: *Stable mappings and their singularities*. Graduate Texts in Mathematics, 14, 1973
- [8] N.H. Kuiper, C^1 -isometric embeddings, *I Proc. Koninkl. Nederl. Ak. Wet. A*, **58**(1955) 545–556.
- [9] J. Nash, C^1 -isometric embeddings, *Annals of Math.* **60**(1954) 383–396.
- [10] D. Spring: *Convex Integration Theory: Solutions to the h-principle in geometry and topology*. Monographs in Mathematics, 92. Birkhäuser-Verlag, 1998.

STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 203, B.T. ROAD, CALCUTTA 700108, INDIA., E-MAIL: MAHUYA@ISICAL.AC.IN